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Inverse problem of the variational calculus for higher KdV equations

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Abstract. It is shown that the usual Hamilton's variational principle supplemented by the methodology of the integer-programming problem can be used to construct expressions for the Lagrangian densities of higher KdV fields. This is demonstrated with special emphasis on the second and third members of the hierarchy. However, the method is general enough for applications to equations of any order. The expressions for Lagrangian densities are used to calculate results for Hamiltonian densities that characterize Zakharov-Faddeev-Gardner equation.

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Three decades have passed when the Korteweg-de Vries (KdV) equation

$$u_t = -u_{xxx} + 6uu_x \tag{1}$$

was interpreted by Zakharov and Faddeev [1] and by Gardner [2] as a completely integrable Hamiltonian system in an infinite dimensional phase space. The Hamiltonian form of (1) is given by

$$u_t = \partial_x \frac{\delta \mathcal{H}}{\delta u} \tag{2}$$

with $\partial_x = \partial/\partial x$, the Hamiltonian operator and $\mathcal{H} = u^3 - u_x^2/2$, the Hamiltonian density. Here $u = u(x,t)$ and subscripts of u refer to the corresponding partial derivatives. The variational or Euler derivative

$$\frac{\delta}{\delta x} = \frac{\partial}{\partial u} - \frac{\partial}{\partial x}\frac{\partial}{\partial u_x} + \frac{\partial^2}{\partial x^2}\frac{\partial}{\partial u_{2x}} - \cdots \tag{3}$$

Hierarchies of infinitely many commuting vector fields and constants of the motion in involution for the KdV equation were constructed by Lax [3] and by Gel'fand and Dikii [4] using the equation for the squared eigenfunction of the Schrödinger operator. The equations in the hierarchy, often called the higher KdV equations, exhibit the same Hamiltonian structure as given in (2). The object of the present work is to solve the inverse problem of variational calculus for equations in the KdV hierarchy and thus derive a method to construct expressions for Hamiltonian densities that characterize the Zakharov-Faddeev-Gardner equation in (2).

All problems of variational methods fall into one of the two groups. (i) The direct problem, where one first assigns a Lagrangian to the dynamical system and then computes the equations of motion and, (ii) the inverse problem posed in the opposite direction. Here one begins with the equations of motion and tries to derive the Lagrangian for the system. The representation of a system in terms of Euler-Lagrange equations goes by the name analytic representation [5]. For nonlinear evolution equations the Helmholtz theorem [6] serves as a useful tool to solve the inverse problem and thereby study their analytic representation. Being derivable from the theory of Lie groups this theorem is both algebraic and geometric in nature. For example, the selfadjointness of the Frechet derivative

$$D_P(Q) = \frac{\mathrm{d}}{\mathrm{d}\epsilon} P[u + \epsilon Q[u]\,]\,|_{\epsilon=0} \tag{4}$$

guarantees the existence of the Lagrangian while the explicit expression for the velocity independent part of the Lagrangian density, $\mathcal{L}_2$ is constructed by using the homotopy formula

$$\mathcal{L}_2[u] = \int_0^1 u\, P[\lambda u]\, \mathrm{d}\lambda. \tag{5}$$

Here P stands for the Euler-Lagrange expression of the variational problem. In the following we use (4) and (5) to deal with the KdV equations.

One common trick to put a single evolution equation into the variational form is to replace u by a potential

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function w defined by

$$w(x,t) = \int_x^\infty \mathrm{d}y\, u(y,t) \qquad (6)$$

such that

$$w_x(x,t) = -u(x,t). \qquad (7)$$

Note that this trick works only for equations which are of odd order in space derivatives and, fortunately for us, all higher KdV equations belong to this class. Using (7) in (1) we get

$$w_{xt} + w_{4x} + 6w_x w_{2x} = 0. \qquad (8)$$

From (8) the Euler-Lagrange expression

$$P[w] = w_{4x} + 6w_x w_{2x}, \qquad (9)$$

using (4), gives the self adjoint operator

$$D_P = D_x^4 + 6w_{2x}D_x + 6w_x D_x \qquad (10)$$

proving the existence of the Lagrangian for (1). The Lagrangian density $\mathcal{L}_2$ for $P[w]$ in (9) can be calculated from (5) to give

$$\mathcal{L}_2 = \frac{1}{2}ww_{4x} + 2ww_x w_{2x}. \qquad (11)$$

The total Lagrangian density for (8) is obtained by adding a velocity, (w_t), dependent part, $\mathcal{L}_1 = w_x w_t/2$ to $\mathcal{L}_2$ and we have

$$\mathcal{L}_h = \frac{1}{2}w_x w_t + \frac{1}{2}ww_{4x} + 2ww_x w_{2x}. \qquad (12)$$

The expression in (12) is of first order in time and fourth order in space. It is of interest to note that the Lagrangian density for (1) or (8) can also be computed without making use of the homotopy formula (5). To see this we integrate (8) with respect to x and write

$$w_t + w_{3x} + 3w_x^2 = 0. \qquad (13)$$

Here we have used the boundary conditions $u(x,t) \to 0$ as $x \to \pm\infty$. It is easy to express (13) in the variational form [7]

$$\delta \int_{t_1}^{t_2} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}x\, \mathcal{L}\left(w_t, w_x, w_{xx}\right) = 0 \qquad (14)$$

with the Lagrangian density

$$\mathcal{L}_v = \frac{1}{2}w_t w_x - \frac{1}{2}w_{2x}^2 + w_x^3. \qquad (15)$$

The subscripts h and v of $\mathcal{L}$ refer to Lagrangian densities obtained from the homotopy formula and direct use of variational principle. As opposed to the result for $\mathcal{L}_h$, the expression for $\mathcal{L}_v$ is only of second order in space derivative. However, $\mathcal{L}_h$ and $\mathcal{L}_v$ are equivalent because they differ by a gauge function and one can write

$$\mathcal{L}_h = \mathcal{L}_v + \partial_x \left(\frac{1}{2}ww_{3x} - \frac{1}{2}w_x w_{2x} - ww_x^2\right). \qquad (16)$$

It is easy to verify that both $\mathcal{L}_h$ and $\mathcal{L}_v$ when substituted in the appropriate Euler-Lagrange equations reproduce the evolution equations in (13). Moreover, both of them lead to the canonical momentum density

$$\pi = \frac{1}{2}w_x. \qquad (17)$$

This equation cannot be inverted for the velocity w_t implying that the Lagrangian densities are degenerate [8]. Therefore one must use the Dirac's theory of constraints [9] to obtain the total Hamiltonian density given by

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1. \qquad (18)$$

Here $\mathcal{H}_0$ is the free part of $\mathcal{H}$ determined by the usual Legendre map and evaluation of the expression for $\mathcal{H}_1$ requires the explicit use of Dirac's theory. It is of interest to note that the free part $\mathcal{H}_0$ only in conjunction with equations (2, 3, 6) reproduces the KdV equation in (13) written [10] in terms of w. The Hamiltonian densities $\mathcal{H}_{0h}$ and $\mathcal{H}_{0v}$ corresponding to $\mathcal{L}_h$ and $\mathcal{L}_v$ can be obtained as

$$\mathcal{H}_{0h} = -\frac{1}{2}ww_{4x} - 2ww_x w_{2x} \qquad (19)$$

and

$$\mathcal{H}_{0v} = \frac{1}{2}w_{2x}^2 - w_x^3. \qquad (20)$$

We note that both $\mathcal{H}_{0h}$ and $\mathcal{H}_{0v}$ have been obtained from gauge equivalent Lagrangians. Despite that $\mathcal{H}_{0h}$ when substituted in (2) does not give the equation of motion (8). However (8) can be obtained by using $\mathcal{H}_{0v}$ in (2) expressed in terms of w.

In view of the above we feel that derivation of a physico-mathematical method other than that based on Helmholtz theorem to construct expressions for free Hamiltonian densities associated with the equations in the entire KdV hierarchy is a problem of considerable significance. We shall proceed by solving the inverse problem and constructing expressions for Lagrangian densities. In principle, there should not be any difficulty to do so, because the Lagrangian always exists for one dimensional cases [11]. However, we find that it is a nontrivial algebraic problem to recast the higher KdV equations in the variational form. This is true even for the first member of hierarchy. In this work we present a general method to construct the Lagrangian densities for higher KdV equations. Our approach to the problem is based on the fact that some of the terms in a given higher equation can be put in the variational form while some others can not be. The first class of terms determines the dimension of the Lagrangian densities such that we could add a linear combination of dimensionally correct terms formed from the products of the derivatives of w. Admittedly, the additive terms are expected to take care of the terms which could not be expressed in the variational form. An added advantage of our method is that it can generate all gauge equivalent Lagrangians in a rather natural way. Further, for any equation the theory can predict the existence of a still higher order Lagrangian than that enters into the usual variational formulation. As opposed to the results obtained from (5) the Hamiltonians constructed from these Lagrangians are consistent with Zakharov-Faddeev-Gardner equation in (2). We begin by demonstrating this with special emphasis on the second member of the KdV hierarchy and then test the effective applicability of our method going to the next member. These case studies exhibit all mathematical complications that need to be considered for any higher KdV equation.

The equation for the second member of the KdV hierarchy is given by

$$u_t = u_{7x} + 14uu_{5x} + 42u_x u_{4x} + 70u_{2x}u_{3x} + 70u^2 u_{3x} + 280uu_x u_{2\ x} + 70u_x^3 + 140u_x u^3. \quad (21)$$

In terms of $w(x,t)$, equation (21) reads

$$w_t = w_{7x} - 14w_x w_{5x} - 28w_{2x}w_{4x} - 21w_{3x}^2 + 70w_x^2 w_{3x} + 70w_x\ w_{2x}^2 - 35w_x^4. \quad (22)$$

Multiplying (22) by δw_x and integrating over t and x we write

$$\int_{t_1}^{t_2} \mathrm{d}t \int_{-\infty}^{+\infty} \mathrm{d}x\, [w_t - w_{7x} + 14w_x w_{5x} + 28w_{2x}w_{4x} + 21w_{3x}^2 - 70w_x^2 w_{3x} - 70w_x w_{2x}^2 + 35w_x^4\,]\,\delta w_x = 0. \quad (23)$$

The first, second and last terms in the square bracket of (23) can be put in the variational form as $(1/2)\delta\int_{t_1}^{t_2}\mathrm{d}t\int_{-\infty}^{+\infty}\mathrm{d}x w_t w_x$, $-(1/2)\delta\int_{t_1}^{t_2}\mathrm{d}t\int_{-\infty}^{+\infty} w_{4x}^2 \mathrm{d}x$ and $(1/5)\delta\int_{t_1}^{t_2}\mathrm{d}t\int_{-\infty}^{+\infty} w_x^5 \mathrm{d}x$ respectively. Thus the Lagrangian density for the evolution equation in (22) has a dimension $[L^{-10}]$ since w is of dimension $[L^{-1}]$. We venture to suggest that the other terms in $\mathcal{L}$ can be accounted for by a dimensionally correct linear combination of the form $\sum \alpha_i w_x^p w_{2x}^q\ w_{3x}^r w_{4x}^s$. Here, the summation is taken over all possible combination of p, q, r and s (integers) such that

$$2p + 3q + 4r + 5s = 10 \quad (24)$$

subject to the constraint

$$p, q, r, s \geq 0. \quad (25)$$

The structure of the equation in (24) imposes a further constraint on the unknowns and we have

$$p \leq 5,\ q < 4,\ r < 3,\ s \leq 2. \quad (26)$$

Table 1. Allowed values of p, q, r and s.

Set	p	q	r	s
1	0	2	1	0
2	1	0	2	0
3	2	2	0	0
4	3	0	1	0
5	1	1	0	1

Ideally, given the constraints in (25) and (26), determination of the values of p, q, r and s involves quite a complicated integer programming problem [12]. However, in the present case one can put forth some plausible arguments to simplify the problem considerably. We point out that the terms with $p = 5$ and $s = 2$ are identical with those already expressed in the variational form. In view of this the equality signs in (26) can be deleted. If we now write two equations from (24) using $s = 0$ and 1, these will impose further restrictions on (26) giving finally

$$p \leq 3,\ q \leq 2,\ r \leq 2,\ s \leq 1. \quad (27)$$

The set of values for p, q, r and s are given in Table 1. The set 1 gives rise to a term which is a perfect differential. Being a gauge term, it can be omitted for all future consideration. Thus, we can write the Lagrangian density in the form

$$\mathcal{L} = \frac{1}{2}w_t w_x + \frac{1}{2}w_{4x}^2 + \alpha_1 w_x w_{3x}^2 + \alpha_2 w_x^2 w_{2x}^2 + \alpha_3 w_x^3 w_{3x} + \alpha_4 w_x w_{2x} w_{4x} + 7w_x^5. \quad (28)$$

We now demand that $\mathcal{L}$ in (28) when substituted in the appropriate Euler-Lagrange equation

$$\frac{\delta\mathcal{L}}{\delta w} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial w_t}\right) = 0 \quad (29)$$

should reproduce the evolution equation in (22). This gives $\alpha_1 = \alpha_4 + 7$ and $\alpha_2 = 3\alpha_3 + 35$. Thus the calculated Lagrangian density will depend on the choices of the values α_3 and α_4. If we write three different expressions for $\mathcal{L}$, namely, $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ corresponding to arbitrary values $(1,1)$, $(2,2)$ and $(3,3)$ for the pair $(\alpha_3,\ \alpha_4)$ then it is easy to see that these Lagrangian densities differ by a total derivative and we have

$$\mathcal{L}_3 - \mathcal{L}_2 = \mathcal{L}_2 - \mathcal{L}_1 = \frac{\mathrm{d}}{\mathrm{d}x}\left(w_x^3 w_{2x} + w_x w_{2x} w_{3x} - \frac{1}{3}w_{2x}^3\right). \quad (30)$$

The Lagrangian density in (28) for chosen values of α's is of order 4. Using the identity

$$w_{4x}^2 = \frac{\mathrm{d}}{\mathrm{d}x}(w_{3x}w_{4x}) - w_{3x}w_{5x} \quad (31)$$

we can construct a fifth-order Lagrangian density given by

$$\mathcal{L}' = \frac{1}{2}w_t w_x - \frac{1}{2}w_{3x}w_{5x} + 8w_x w_{3x}^2 + 38w_x^2 w_{2x}^2 + w_x^3 w_{3x} + w_x w_{2x} w_{4x} + 7w_x^5, \quad (32)$$

with $(\alpha_3, \alpha_4) = (1,1)$.

In writing (32) we have omitted the total derivative term. One can easily verify that $\mathcal{L}'$ *via* (29) also yields the evolution equations. It is important to note that $\mathcal{L}'$ is of order 5 and further enhancement of order is not possible.

Both $\mathcal{L}$ and $\mathcal{L}'$ can be Hamiltonized to write

$$\mathcal{H}_0 = -\frac{1}{2}w_{4x}^2 - 8w_x w_{3x}^2 - 38w_x^2 w_{2x}^2 - w_x^3 w_{3x} - w_x w_{2x} w_{4x} - 7w_x^5 \tag{33}$$

and

$$\mathcal{H}_0' = \frac{1}{2}w_{3x}w_{5x} - 8w_x w_{3x}^2 - 38w_x^2 w_{2x}^2 - w_x^3 w_{3x} - w_x w_{2x} w_{4x} - 7w_x^5. \tag{34}$$

The expressions for $\mathcal{H}_0$ and $\mathcal{H}_0'$ in conjunction with (2) produce the same evolution equation as given in (22).

In terms of the above procedure we have calculated $\mathcal{L}$ for third member in the hierarchy given by

$$\begin{aligned} u_t = u_{9x} &- 252\, u_{3x}u_{4x} - 168\, u_{2x}u_{5x} - 72\, u_x u_{6x} \\ &- 18\, uu_{7x} + 1302\, u_x u_{2x}^2 + 966\, u_x^2 u_{3x} \\ &+ 1260\, uu_{2x}u_{3x} + 756\, uu_x u_{4x} + 126\, u^2 u_{5x} \\ &- 1260\, uu_x^3 - 2520\, u^2 u_x u_{2x} - 420\, u^3 u_{3x} + 630\, u^4 u_x. \end{aligned} \tag{35}$$

The general expression for $\mathcal{L}$ can be written as

$$\begin{aligned} \mathcal{L} = \frac{1}{2}w_t w_x &- \frac{1}{2}w_{5x}^2 - 21w_x^6 + \alpha_1 w_{2x}^4 + \alpha_2 w_{3x}^3 \\ &+ \alpha_3 w_{2x}^2 w_{5x} + \alpha_4 w_{2x} w_{3x} w_{4x} + \alpha_5 w_x w_{4x}^2 \\ &+ \alpha_6 w_x w_{3x} w_{5x} + \alpha_7 w_x w_{2x}^2 w_{3x} + \alpha_8 w_x^2 w_{3x}^2 \\ &+ \alpha_9 w_x^2 w_{2x} w_{4x} + \alpha_{10} w_x^3 w_{2x}^2 + \alpha_{11} w_x^3 w_{5x} + \alpha_{12} w_x^4 w_{3x} \end{aligned} \tag{36}$$

with

$$\alpha_5 - \alpha_6 = 9, \tag{37}$$

$$\alpha_{10} - 4\alpha_{12} = 210, \tag{38}$$

$$\alpha_9 - \alpha_8 - 3\alpha_{11} = 63, \tag{39}$$

$$3\alpha_4 - 6\alpha_2 - 6\alpha_3 - 3\alpha_5 = 33, \tag{40}$$

$$\text{and} \quad 2(3\alpha_1 - \alpha_7 + 3\alpha_8 - \alpha_9) = -273. \tag{41}$$

The relations in (37–41) indicate that seven different α's can be chosen arbitrarily to generate gauge equivalent Lagrangians for the ninth order equation in (35). Note that such freedom was limited only to two α values for the equation in (21). In the present case for $\alpha_3 = \alpha_4 = \alpha_6 = \alpha_7 = \alpha_9 = \alpha_{11} = \alpha_{12} = 0$, we have found

$$\mathcal{L} = \frac{1}{2}w_t w_x - \frac{1}{2}w_{5x}^2 - 21w_x^6 + \frac{35}{2}w_{2x}^4 - 10w_{3\ x}^3 + 9w_x w_{4x}^2 - 63w_x^2 w_{3x}^2 + 210w_x^3 w_{2x}^2 \tag{42}$$

and

$$\mathcal{H}_0 = \frac{1}{2}w_{5x}^2 + 21w_x^6 - \frac{35}{2}w_{2x}^4 + 10w_{3x}^3 - 9w_x w_{4x}^2 + 63w_x^2 w_{3x}^2 - 210w_x^3 w_{2x}^2. \tag{43}$$

It is easy to verify that the consistency of equations (2, 43) gives (35) expressed in terms of $w(x,t)$.

We conclude by noting that the present work starts with higher KdV equations and derives a method to construct expressions for corresponding Lagrangian densities which need not be unique. These equivalent Lagrangian densities result in compatible Hamiltonian densities for the theory of Zakharov *et al.* [1,2].

The canonical formulation for higher order nonsingular or nondegenerate Lagrangian was derived by Ostrogradski [13] more than 150 years ago. This approach has recently been adapted for the field theory. In the full form of the field theoretic generalization [14] of Ostrogradski formalism one would require to calculate canonical momenta corresponding to w and its derivatives for calculating $\mathcal{H}_0$. Since the evolution equations are of first order in time we could proceed by working out the momentum density conjugate to w alone.

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